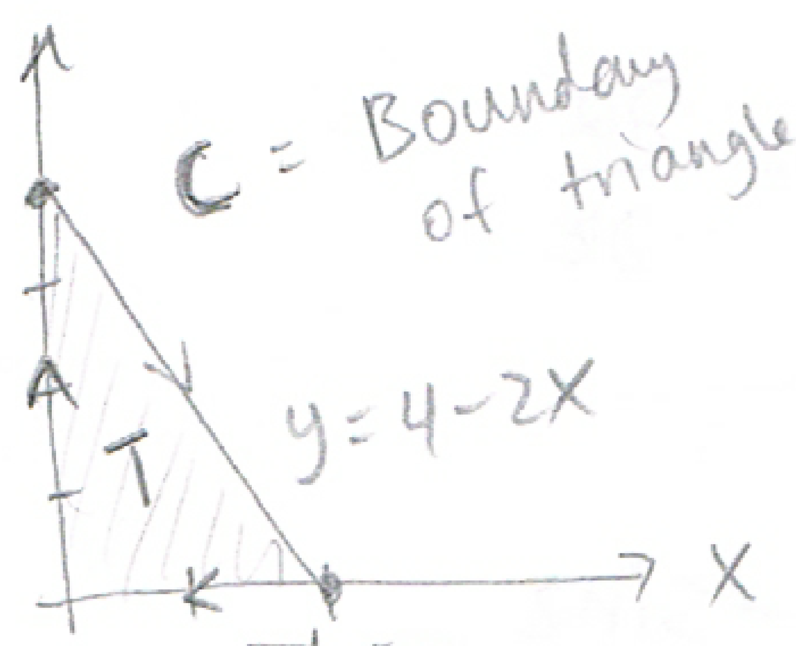
(Solution)

Math 2E Quiz 8 Afternoon - May 19th

Please write your name and ID on the front.

Show all of your work, and simplify all your answers. *There is a question on the back side.

1. Let $\mathbf{F}(x,y) = \langle \underbrace{y\cos y + \frac{y^2}{2}}_{P=}, \underbrace{x\cos y - xy\sin y}_{Q=} \rangle$. Let C be the boundary of the triangle with vertices at $(0,0)$, $(0,4)$, and $(2,0)$ with clockwise orientation. Compute $\oint_C \mathbf{F}\cdot d\mathbf{r}$.



C = Boundary of triangle, T

- Components of $\vec{F}$ are smooth so we can use Green's Theorem. Here, $P = y\cos y + \frac{y^2}{2}$, and $Q = x\cos y - xy\sin y$.

This ☆ Orientation is Negative!! +1

Defn: $\oint_C \vec{F}\cdot d\vec{r} \overset{\text{orientation}}{=} - \oint_{-C} P\,dx + Q\,dy \overset{\text{Green's}}{=} - \iint_T \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

↖ In other words, $\text{curl}(\vec{F})$, or $\nabla\times\vec{F}$.

Here, $\frac{\partial Q}{\partial x} = \cos y - y\sin y$

$\frac{\partial P}{\partial y} = \cos y - y\sin y + y$ $\Rightarrow \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -y$.

Note: $-C$ has positive orientation, so we apply Green's to $-C$.

+2 +2

So, $\oint_C \vec{F}\cdot d\vec{r} = -\oint_{-C} P\,dx + Q\,dy \overset{\text{Green's}}{=} - \iint_T \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

$= -\int_{x=0}^{2}\int_{y=0}^{4-2x} -y\,dy\,dx$ +2

$= \int_{x=0}^{2} \frac{y^2}{2}\Big|_0^{4-2x} dx = \int_0^2 \frac{16-16x+4x^2}{2}\,dx$ +2

$= 8x - 4x^2 + \frac{2x^3}{3}\Big|_0^2 = \cancel{16} - \cancel{16} + \frac{16}{3} = \boxed{\frac{16}{3}}$ +1

Incorrect Sign: -2

P? Q?

2. Let $\mathbf{F}(x, y) = < x^2y^5,\ ax^by^c >$ where a, b, c are real numbers.

a) Find values of a, b, c that make the vector field $\mathbf{F}$ conservative.
(You may assume that since polynomial functions are smooth on all of $\mathbb{R}^2$, the components of $\mathbf{F}$ are continuously differentiable, in fact, smooth, on all of $\mathbb{R}^2$.) (Smooth means infinitely differentiable).

The components are smooth ✓ so to be conservative, need that

+1 $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ (i.e. that $\text{curl}(\vec{F}) = \nabla \times \vec{F} = \vec{0}$). ($\vec{0}$ denotes the zero vector, $0\hat{i} + 0\hat{j} + 0\hat{k}$).

$$\left.\begin{aligned} \frac{\partial Q}{\partial x} &= abx^{b-1}y^c \\ \frac{\partial P}{\partial y} &= 5x^2y^4 \end{aligned}\right\}$$

+1 +1 +1

$c = 4,\ b = 3,\ a = \frac{5}{3}$.

for y^4 for x^2 for coefficient

b) Let C be the path $x = \sin t, y = 1 + \sin 2t$ from $-\pi/2 \le t \le \pi/2$. Compute $\int_C \mathbf{F} \cdot \mathbf{dr}$.
(Plug in the values of a, b, c you found in part (a) into the definition of $\mathbf{F}$).

Now, $\vec{F}(x,y) = < x^2y^5,\ \frac{5}{3}x^3y^4 >$. Since conservative, use Fund. Thm. of Line Integrals.

1) Get f s.t. $\nabla f = \vec{F}$:

- $f(x,y) = \int x^2y^5\,dx + g(y)$

+1 +1

$$= \frac{x^3y^5}{3} + g(y)$$

- $f_y = \frac{5}{3}x^3y^4 + g_y(y) \equiv \frac{5}{3}x^3y^4 \Rightarrow g_y(y) = 0;\ g(y) \equiv K$ (constant) +1

Hence, $\boxed{f(x,y) = \frac{x^3y^5}{3} + K}$

2) Apply Fund. Thm: Final Pt, $t = \pi/2$: $(\sin\frac{\pi}{2}, 1 + \sin\pi) = (1,1)$. +1

Initial Pt, $t = -\pi/2$: $(\sin(-\frac{\pi}{2}), 1 + \sin(-\pi)) = (-1,1)$.

$$\oint_C \vec{F}\cdot d\vec{r} = f(1,1) - f(-1,1) = \left(\frac{1}{3} + \cancel{K}\right) - \left(\frac{(-1)^3}{3} + \cancel{K}\right)$$

+1

$$= \boxed{2/3}$$ +1